Übungsstunde 11:

Themen:

- Definitheit von Matrizen
- Singulärwertzerlegung (SVD) für die Ausgleichsrechnung

Definitheit: Hurwitz-Kriterium / Sylvester-Kriterium

$\underline{\underline{A}}$ symmetrisch, quadratisch ist positiv definit, falls alle führenden Hauptminoren von $\underline{\underline{A}}$ positiv sind.

$$\underline{x}^T \underline{\underline{A}}\, \underline{x} > 0 \quad \forall\, \underline{x} \in \mathbb{R} \setminus \{\underline{0}\}$$

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1i} \\ \vdots & \ddots & \\ a_{i1} & & a_{ii} \end{bmatrix} > 0 \quad \forall i \in [1, n], \quad \underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & \ddots & & \vdots \\ \vdots & & \ddots & \\ a_{1n} & \cdots & & a_{nn} \end{bmatrix}$$

Bsp:

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & -4 \\ 0 & -4 & 3 \end{bmatrix}$$

$$a_1 = \det[2] = \underline{2 > 0}$$

$$a_2 = \det \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} = 16 - 0 = \underline{16 > 0}$$

$$a_3 = \det \begin{bmatrix} \boxed{2^+} & 0^- & 0^+ \\ 0^- & 8^+ & -4^- \\ 0^+ & -4^- & 3^+ \end{bmatrix} = 2 \cdot \det \begin{bmatrix} 8 & -4 \\ -4 & 3 \end{bmatrix} = 2 \cdot (24 - 16)$$
$$= \underline{16 > 0}$$

$\rightsquigarrow$ $\underline{\underline{A}}$ pos. definit!

Vorzeichen $<0, >0, <0, >0 \ldots$
$\rightarrow$ $\underline{\underline{A}}$ neg. definit

Singulärwertzerlegung (SVD)

$$\underline{\underline{A}} = \underline{\underline{U}} \cdot \underline{\underline{S}} \cdot \underline{\underline{V}}^T \quad , \quad \underline{\underline{S}} = \begin{bmatrix} \hat{\underline{\underline{S}}} \\ 0 \end{bmatrix}, \quad \underline{\underline{U}} \,\&\, \underline{\underline{V}} \text{ orthogonal}$$

$$\|\underline{r}\|_2^2 = \|\underline{\underline{A}}\,\underline{x} - \underline{c}\|_2^2 = \|\underline{\underline{U}}\,\underline{\underline{S}}\,\underline{\underline{V}}^T \underline{x} - \underline{c}\|_2^2$$

$$= \|\underline{\underline{U}}(\underline{\underline{S}}\,\underline{\underline{V}}^T \underline{x} - \boxed{\underline{\underline{U}}^T \underline{c}})\|_2^2 \qquad \|\underline{\underline{U}}\|_2^2 = 1, \text{ da } \underline{\underline{U}} \text{ orth.}$$

$$= \|\underline{\underline{S}}\,\underline{\underline{V}}^T \underline{x} - \underbrace{\underline{\underline{U}}^T \underline{c}}\|_2^2$$

$$= \|\underbrace{\underline{\underline{S}}\,\underline{\underline{V}}^T \underline{x} - \underline{d}}_{\begin{bmatrix} \hat{S} V^T x \\ 0 \end{bmatrix} - \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}}\|_2^2 = \underbrace{\|\hat{\underline{\underline{S}}}\,\underline{\underline{V}}^T \underline{x} - \underline{d_0}\|_2^2}_{\overset{!}{=} 0} + \underbrace{\|d_1\|_2^2}_{= \|r\|_2^2 = \text{Fehler}}$$

$$\underline{\hat{\underline{S}}\,\underline{\underline{V}}^T \underline{x} = \underline{d_0}}$$

2 Möglichkeiten!

$\Rightarrow$ $\hat{\underline{\underline{S}}}$ inv.: $\quad \underline{x} = \underline{\underline{\underline{V}}\,\hat{\underline{\underline{S}}}^{-1}\,\underline{d_0}}$

$\hat{\underline{\underline{S}}}$ nicht inv.: $\quad \underline{x} = \underline{\underline{\underline{V}}\,\hat{\underline{\underline{S}}}^{+}\,\underline{d_0}}$ $\qquad \hat{S}^+$ ist die $\boxed{\text{Pseudo-Inverse}}$ von $\hat{S}$!

$\underline{\underline{U}}$: EV von $\underline{\underline{A}} \cdot \underline{\underline{A}}^T$

$\underline{\underline{V}}$: EV von $\underline{\underline{A}}^T \cdot \underline{\underline{A}}$

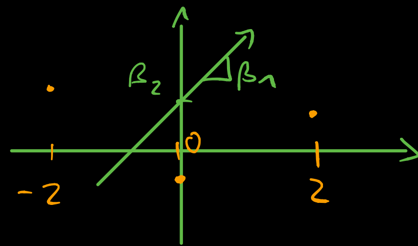
Der Grösse nach sortiert!

S: $\text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_p})$ EW von $\underline{\underline{A}}\,\underline{\underline{A}}^T$ o. $\underline{\underline{A}}^T \underline{\underline{A}}$, $p = \min\{m, n\}$

$$\underline{v}^{(i)} = \frac{\underline{\underline{A}}^T u^{(i)}}{\sigma^{(i)}}, \quad \underline{u}^{(i)} = \frac{\underline{\underline{A}}\, v^{(i)}}{\sigma^{(i)}}, \quad \sigma^{(i)} = \sqrt{\lambda_i}$$

Bsp.: Prüfung HS 16

$y = \beta_1 x + \beta_2 \iff$



x_i	2	0	-2
y_i	$\sqrt{6}$	$-\sqrt{6}$	$2\sqrt{6}$

$\sum_{i=1}^{3} |y_i - (\beta_1 x_i + \beta_2)|^2$ soll minimiert werden

$$\underline{\underline{A}}\,\underline{\beta} = \underline{c}$$

$$\underline{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \quad \underline{c} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \sqrt{6} \\ -\sqrt{6} \\ 2\sqrt{6} \end{bmatrix}, \quad \underline{\underline{A}} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -2 & 1 \end{bmatrix}$$

$$\underline{\underline{A}} = \underline{\underline{U}} \cdot \underline{\underline{S}} \cdot \underline{\underline{V}}^T$$

$$U: \underline{\underline{A}}\,\underline{\underline{A}}^T = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -3 \\ 1 & 1 & 1 \\ -3 & 1 & 5 \end{bmatrix}$$

$$\text{EW: } \det(\underline{\underline{A}}\,\underline{\underline{A}}^T - \lambda \underline{\underline{I}}) \overset{!}{=} 0 = \det \begin{bmatrix} 5-\lambda & 1 & -3 \\ 1 & 1-\lambda & 1 \\ -3 & 1 & 5-\lambda \end{bmatrix}$$

$$= (5-\lambda) \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 5-\lambda \end{bmatrix} - \det \begin{bmatrix} 1 & -3 \\ 1 & 5-\lambda \end{bmatrix} - 3 \det \begin{bmatrix} 1 & -3 \\ 1-\lambda & 1 \end{bmatrix}$$

$$= (5-\lambda)\left[(1-\lambda)(5-\lambda) - 1\right] - (5-\lambda+3) - 3(1+3-3\lambda)$$

$$= (1-\lambda)(5-\lambda)^2 - (5-\lambda) - (8-\lambda) - (12-9\lambda)$$

$$= (1-\lambda)(25-10\lambda+\lambda^2) + 11\lambda - 25$$

$$= -\lambda^3 + 11\lambda^2 - 24\lambda \overset{!}{=} 0$$

$$= -\lambda(\lambda^2 \underbrace{-11}_{+}\lambda \underbrace{+24}_{\cdot}) = -\lambda(\lambda-3)(\lambda-8)$$

$$\Rightarrow \quad \underline{\lambda_1 = 8}, \ \underline{\lambda_2 = 3}, \ \underline{\lambda_3 = 0}$$

$$\Rightarrow \quad \sigma_1 = \sqrt{8} = \underline{2\sqrt{2}}, \ \sigma_2 = \underline{\sqrt{3}}, \ \sigma_3 = \underline{0}$$

$$\Rightarrow \quad \underline{\underline{S}} = \underline{\begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}}$$

EV:

$$\lambda_1 = 8: \quad \left[\begin{array}{ccc|c} -3 & 1 & -3 & 0 \\ 1 & -7 & 1 & 0 \\ -3 & 1 & -3 & 0 \end{array}\right] \xrightarrow{G.} \left[\begin{array}{ccc|c} 1 & -7 & 1 & 0 \\ 0 & -20 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_3 = s \in \mathbb{R} \\ x_2 = 0 \\ x_1 = -s \end{array}$$

$$\Rightarrow \quad E_8 = \operatorname{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\} = \underline{\operatorname{span}\left\{\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}}$$

$$\lambda_2 = 3: \quad \left[\begin{array}{ccc|c} 2 & 1 & -3 & 0 \\ 1 & -2 & 1 & 0 \\ -3 & 1 & 2 & 0 \end{array}\right] \xrightarrow{G} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_3 = w \in \mathbb{R} \\ x_2 = w \\ x_1 = w \end{array}$$

$$\Rightarrow \quad E_3 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\} = \underline{\operatorname{span}\left\{\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}}$$

$$\lambda_3 = 0: \quad \left[\begin{array}{ccc|c} 5 & 1 & -3 & 0 \\ 1 & 1 & 1 & 0 \\ -3 & 1 & 5 & 0 \end{array}\right] \xrightarrow{G.} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_3 = t \in \mathbb{R} \\ x_2 = -2t \\ x_1 = t \end{array}$$

$$\Rightarrow E_0 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\} = \underline{\operatorname{span}\left\{\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}}$$

$$\Rightarrow \underline{\underline{U}} = \underline{\frac{1}{\sqrt{6}}\begin{bmatrix} -\sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \\ \sqrt{3} & \sqrt{2} & 1 \end{bmatrix}}$$

$$\underline{v}^{(1)} = \frac{\underline{\underline{A}}^T \underline{u}^{(1)}}{\sigma^{(1)}} = \frac{\begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}{2\sqrt{2}} = \frac{1}{4}\begin{bmatrix} -4 \\ 0 \end{bmatrix} = \underline{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}$$

$$\underline{v}^{(2)} = \frac{\underline{\underline{A}}^T \underline{u}^{(2)}}{\sigma^{(2)}} = \frac{\begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{3}} = \frac{1}{3}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \underline{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$\Rightarrow \underline{\underline{V}} = \underline{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}$$

⚠ In SVD wird V^T benutzt!

$$\underline{\underline{A}}\,\underline{x} = \underline{\underline{U}}\,\underline{\underline{S}}\,\underline{\underline{V}}^T \underline{x} = \underline{c}$$

$$\Leftrightarrow \quad \underline{\underline{S}}\,\underline{\underline{V}}^T \underline{x} = \underline{\underline{U}}^T \underline{c} = \underline{d} = \begin{bmatrix} d_0 \\ \underline{d_1} \end{bmatrix}$$

$$\Rightarrow \quad \underline{\underline{\hat{S}}}\,\underline{\underline{V}}^T x = \underline{d_0}$$

$$\underline{d} = \underline{\underline{U}}^T \underline{c} = \frac{1}{\sqrt{6}} \begin{bmatrix} -\sqrt{3} & 0 & \sqrt{3} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \\ * & * & * \end{bmatrix} \begin{bmatrix} \sqrt{6} \\ -\sqrt{6} \\ 2\sqrt{6} \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ 2\sqrt{2} \\ * \end{bmatrix} \Big\} \underline{d}_0$$

$$\begin{matrix} 1 & -2 & 1 \end{matrix} \qquad 5 \;\; \Big\} \underline{d}_1 \rightarrow \text{Residuum}$$

$$\Rightarrow \quad \underline{x} = \underline{\underline{V}}\, \hat{\underline{\underline{S}}}^{-1} \underline{d}_0 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 2\sqrt{2} \end{bmatrix} = \underline{\underline{\begin{bmatrix} -\frac{\sqrt{3}}{2\sqrt{2}} \\ \frac{2\sqrt{2}}{\sqrt{3}} \end{bmatrix}}} \begin{matrix} \beta_1 \\ \beta_2 \end{matrix}$$

$$\hat{\underline{\underline{S}}}^{-1} = \frac{1}{\det(\hat{S})} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix} = \underline{\begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix}}$$

Allgemein: $$\hat{\underline{\underline{S}}}^{-1} = \begin{bmatrix} \frac{1}{\sigma_1} & & & \emptyset \\ & \frac{1}{\sigma_2} & & \\ & & \ddots & \\ \emptyset & & & \frac{1}{\sigma_n} \end{bmatrix}$$

$$\hat{\underline{\underline{S}}}^{+} = \begin{bmatrix} \frac{1}{\sigma_1} & & & & & \emptyset \\ & \ddots & & & & \\ & & \frac{1}{\sigma_k} & & & \\ & & & 0 & & \\ & & & & 0 & \\ \emptyset & & & & & 0 \end{bmatrix}$$

Zusammenhang zwischen U & V:

$$\underline{\underline{A}}^T \underline{\underline{A}}\, \underline{v}^{(i)} = \lambda\, \underline{v}^{(i)}$$

$$(\underline{\underline{A}}\, \underline{\underline{A}}^T) \underbrace{\underline{\underline{A}}\, \underline{v}^{(i)}}_{\tilde{\underline{u}}^{(i)}} = \lambda \underbrace{\underline{\underline{A}}\, v^{(i)}}_{\tilde{\underline{u}}^{(i)}} = \lambda\, \tilde{u}^{(i)} \qquad \Rightarrow \quad \underline{u}^{(i)} = \frac{\tilde{\underline{u}}^{(i)}}{\sigma^{(i)}} = \frac{\underline{\underline{A}}\, \underline{v}^{(i)}}{\sigma^{(i)}}$$

$$\underline{\underline{A}}\, \underline{\underline{A}}^T \underline{u}^{(i)} = \lambda\, \underline{u}^{(i)}$$

$$(\underline{\underline{A}}^T \underline{\underline{A}}) \underbrace{\underline{\underline{A}}^T \underline{u}^{(i)}}_{\tilde{\underline{v}}^{(i)}} = \lambda \underbrace{\boxed{\underline{\underline{A}}^T \underline{u}^{(i)}}}_{\tilde{\underline{v}}^{(i)}} = \lambda\, \tilde{\underline{v}}^{(i)} \qquad \Rightarrow \quad \underline{v}^{(i)} = \frac{\tilde{\underline{v}}^{(i)}}{\boxed{\sigma^{(i)}}}$$

Wollen wieder Länge 1!

$$\|\underline{\underline{A}}^T\underline{u}\|_2 = \|(\underline{\underline{U}}\,\underline{\underline{S}}\,\underline{\underline{V}}^T)^T\underline{u}\|_2 = \|\underline{\underline{V}}\,\underline{\underline{S}}^T\underline{\underline{U}}^T\underline{u}\|_2$$

} U & V orthogonal → verändern Länge nicht!

$$= \|\underline{\underline{S}}^T\underline{u}\|_2$$

S diagonal!

$$= \|\sigma^{(i)}\underline{u}^{(i)}\|_2 = \boxed{\|\sigma^{(i)}\|_2}$$

↳ Ergebnis hat also Länge der Singulärwerte

$$\|\underline{\underline{A}}\,\underline{v}\|_2 = \dots$$